A General Mapping Technique for Fourier Transform Computation in Nonlinear Circuit Analysis

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***Abstract*—A mapping technique that can handle any number of fundamental frequencies in multitone nonlinear circuit analysis is presented. In this technique, almost-periodic spectrum truncated using the box scheme is mapped onto an equivalent periodic spectrum which is dense with no missing harmonic. The Fourier transform (or its inverse) is then implemented by a single one-dimensional fast Fourier transform. Characteristics of the mapping technique are illustrated by some results. Due to its good combination of flexibility and speed, this mapping technique should be considered as an alternative to the multidimensional discrete Fourier transform in general-purpose harmonic-balance simulators.**

***Index Terms*—Almost-periodic signals, Fourier transform computation, harmonic balance, mapping techniques.**

I. Introduction

In harmonic-balance (HB) simulation of nonlinear microwave circuits, a Fourier transform is needed to carry out the conversion between time and frequency domains. For periodic circuits, the traditional discrete Fourier transform (DFT) is invariably employed. The DFT is usually implemented by the efficient fast Fourier transform (FFT) algorithm. In multitone problems, the signals become almost-periodic [1]. The most important transforms that have been considered for this situation are the multidimensional discrete Fourier transform (MDFT), the almost-periodic Fourier transforms (APFT's), and mapping techniques [2]. In the latter, almost-periodic spectrum is mapped onto an equivalent periodic spectrum, which can then be handled by the DFT.

Although APFT's are more flexible with respect to spectrum truncation, only the MDFT and mapping techniques can be efficiently implemented by the FFT. The MDFT can be directly formulated for any number of fundamental frequencies [3]. However, mapping techniques are presently available only for two fundamental frequencies [4], [5].

This letter presents a mapping technique suitable for any number of fundamental frequencies. The original almost-periodic spectrum truncated using the box scheme [5] is mapped onto an equivalent dense periodic spectrum with no missing harmonic. The Fourier transform, or its inverse, is then computed by a single one-dimensional FFT. Results are discussed to illustrate the characteristics of this mapping technique.

II. Mapping Techniques

In mapping techniques, the actual fundamental frequencies are replaced by artificial fundamental frequencies so that the original spectrum is mapped onto an equivalent periodic and dense spectrum. By dense, we mean that at most a few harmonics are missing. Signals transformed through the mapping become periodic and, consequently, can have their Fourier coefficients efficiently calculated by the one-dimensional FFT. In addition, as the mapped spectrum is dense, the number of time-domain samples required (which is determined by the periodic spectrum) is not much larger than the theoretical minimum predicted from the original spectrum. The concept of mapping techniques discussed here was originally introduced for diamond truncation with two fundamental frequencies in [4] and later extended to box truncation with the same number of fundamental frequencies [5].

Mapping techniques rely on two basic properties. First, HB can always be formulated to only require computation of Fourier coefficients of signals that can be expressed as $y(t) = f[x(t)]$, where $f(\cdot)$ is an algebraic function and the Fourier coefficients of $x(t)$ are known. When needed, derivatives should be calculated in the frequency domain using the original spectrum. The second property is the frequency independence of Fourier coefficients when fundamental frequencies are incommensurable (i.e., linearly independent over the rationals) [4], [5].

The spectrum in a nonlinear circuit is given by

$$\Lambda = \left\{ \omega \;|\omega = \sum_{l=1}^{d} m_l \lambda_l;\; m_l \in \mathrm{Z}; \;\text{first nonzero } m_l \text{ positive} \right\} \quad (1)$$

where $\{\lambda_1, \cdots, \lambda_d\}$ is the set of fundamental frequencies. The requirement of the λ_l's being incommensurable can be relaxed after spectrum is truncated to a finite set $\Lambda_K = \{\omega_0 = 0, \omega_1, \cdots, \omega_K\}$. Each $\omega_k \in \Lambda_K$ is generated by a vector of indexes $\boldsymbol{m} = (m_1, \cdots, m_d)$. Neglecting effects of aliasing, a Fourier coefficient will depend on actual values in $\{\lambda_1, \cdots, \lambda_d\}$ only if two or more different vectors of indexes yield the same ω_k. Thus, fundamental frequencies need be incommensurable only within Λ_K.

Manuscript received June 20, 1997.

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Publisher Item Identifier S 1051-8207(97)08184-1.

When $\{\lambda_1, \cdots, \lambda_d\}$ is replaced by a set of artificial fundamental frequencies $\{\sigma_1, \cdots, \sigma_d\}$, Λ_K is mapped onto Σ_K. If the artificial fundamental frequencies are also incommensurable within Σ_K, then the mapping is one-to-one and Fourier coefficients remain independent of $\{\sigma_1, \cdots, \sigma_d\}$ as well. In addition to being incommensurable within Σ_K, $\{\sigma_1, \cdots, \sigma_d\}$ should be selected in mapping techniques such that Σ_K is periodic and dense. In case this is possible, $x(t)$ is transformed by the mapping into a periodic signal, time-domain samples of $x(t)$ are efficiently computed from its Fourier coefficients using the FFT, (frequency independent) Fourier coefficients of $y(t) = f[x(t)]$ are efficiently computed by the FFT and then mapped from Σ_K back to Λ_K.

In general, the problem of determining artificial fundamental frequencies having the desired properties is rather complex. Explicit solutions which yield periodic spectra with no missing harmonic are presently available only for diamond and box truncation with two fundamental frequencies.

III. A General Mapping Technique Compatible with Box Truncation

Using box truncation, the truncated frequency set Λ_K becomes

$$\Lambda_K = \left\{ \omega \,|\omega = \sum_{l=1}^{d} m_l\lambda_l;\ m_l \in \mathrm{Z};\ |m_l| \le H_l,\right.$$
$$\left. l = 1, \cdots, d;\ \text{first nonzero } m_l \text{ positive} \right\}. \quad (2)$$

The integer H_l limits the largest harmonic of λ_l which is considered. The number of frequencies in (2) different from zero is equal to

$$K = \tfrac{1}{2}\left[\left(\prod_{l=1}^{d}(2H_l + 1)\right) - 1\right]. \quad (3)$$

A universal sequence of artificial fundamental frequencies is generated using

$$\begin{cases} \sigma_1 = \sigma \\ \sigma_l = (1 + 2H_{l-1})\sigma_{l-1} \qquad l = 2, 3, 4, \cdots \end{cases} \quad (4)$$

for arbitrary $\sigma > 0$. For a given number of fundamental frequencies d, the first d frequencies generated by (4) should be used. From (2) and (4), the mapped spectrum results in

$$\Sigma_K = \{\nu \,|\nu = k\sigma;\ k = 0, 1, \cdots, K\} \quad (5)$$

with K given in (3). Σ_K is, therefore, periodic and dense with no missing harmonic. In addition, the first d σ_l's determined from (4) are incommensurable within the corresponding Σ_K. The order of the harmonic k in (5) is related to $m_1, \cdots, m_d$ in (2) by

$$k = \frac{1}{\sigma}\left|\sum_{l=1}^{d} m_l\sigma_l\right|. \quad (6)$$

This equation implements the mapping between Λ_K and Σ_K. Since σ is arbitrary, it may be set equal to 1 rad/s for simplicity.

In some important situations, fundamental frequencies are originated from independent sources having somewhat different power levels. For instance, this occurs in mixers. To reduce effects of aliasing, the best strategy is to label the original base frequencies in increasing order of power levels. Thus, the higher the power level, the larger the associated σ_l will be. An example of the effectiveness of this strategy will be given in the next section.

The reason for the strategy described above can be understood after an examination of how aliasing is mapped back to the original spectrum. The first fundamental frequency λ_1 is mapped on the first harmonic σ of the mapped periodic spectrum. Therefore, harmonics of λ_1 not accounted for in (2) cause aliasing right in the middle of the original spectrum. On the other hand, harmonics of the last fundamental frequency λ_d not in (2) can only corrupt other harmonics of λ_d. In this case, the aliasing mechanism is similar to that of periodic signals and higher order harmonics of λ_d are corrupted first. Thus, a high power signal will introduce less aliasing errors when its associated fundamental frequency is labeled last.

IV. Results

A set of routines was developed in C to implement the mapping technique. These routines carry out spectrum truncation and reordering as well as direct and inverse Fourier transforms. As usual, the FFT takes into account the fact that time-domain sequences are real [6], [7]. The interface with HB simulators becomes particularly simple if the actual frequencies are ordered according to the mapped spectrum in the simulator. In case a frequency and its corresponding artificial frequency have opposite signs, the related Fourier coefficient should be complex conjugated when moving between real and mapped spectra.

To compare the mapping technique with the FFT-based MDFT, we examined the problem of computing the Fourier coefficients of the signal

$$y(t) = 0.01\ \exp\left(10\ \cos\ \omega_1 t + \cos\ \omega_2 t + \cos\ \omega_3 t\right) \quad (7)$$

where ω_1, ω_2, and ω_3 are arbitrary incommensurable frequencies. For simplicity, $H_1 = H_2 = H_3 = H$ was adopted in (2). In general, different H_l's should be employed to optimize the transform. Fourier coefficients were computed by the mapping technique with ω_1 mapped to the *first* artificial fundamental frequency, by the mapping technique with ω_1 mapped to the *last* artificial fundamental frequency, and using the MDFT.

Fig. 1 shows the computed Fourier coefficient at frequency $\omega_1 + 2\omega_3 - \omega_2$ as a function of H. As expected, the mapping technique is less affected by aliasing when ω_1 is mapped to the last artificial fundamental frequency. In fact, for $H \ge 3$ this strategy performs even better than the MDFT. These results are also displayed on Table I. The dynamic range of the mapping technique is comparable to that of the MDFT, which was assessed in [8] for two fundamental frequencies.

TABLE I
CALCULATED FOURIER COEFFICIENT OF THE SIGNAL DEFINED IN (7) AT $\omega_1 + 2\omega_3 - \omega_2$ AS A FUNCTION OF H (SEE FIG. 1 CAPTION). ALSO LISTED ARE THE OPERATION COUNTS FOR THE MAPPING TECHNIQUE AND FOR THE MDFT

H	Fourier Coeff. Map. Tech.(1)	Fourier Coeff. Map. Tech.(2)	Fourier Coeff. MDFT	Operation Count Map. Tech.	Operation Count MDFT
2	9.59009	28.0088	4.54475	896	4608
3	5.68817	4.12448	4.54475	4608	4608
4	4.50535	4.09848	4.09849	10240	49152
5	4.17471	4.09928	4.09849	22528	49152
6	4.10914	4.09831	4.09849	49152	49152
7	4.09950	4.09831	4.09849	49152	49152
8	4.09842	4.09831	4.09831	106496	491520
9	4.09832	4.09831	4.09831	106496	491520
10	4.09831	4.09831	4.09831	229376	491520

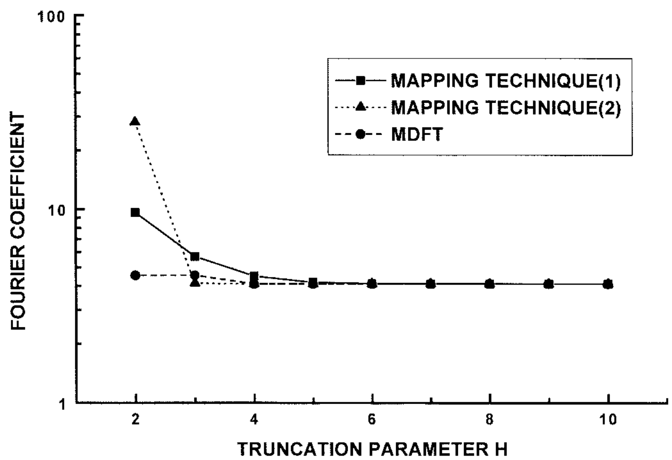

Fig. 1. Fourier coefficient of the signal defined in (7) at frequency $\omega_1 + 2\omega_3 - \omega_2$ as a function of the truncation parameter H. The Fourier coefficient is computed by mapping technique 1)—ω_1 mapped to first artificial fundamental frequency; mapping technique 2)—ω_1 mapped to last artificial fundamental frequency; and MDFT.

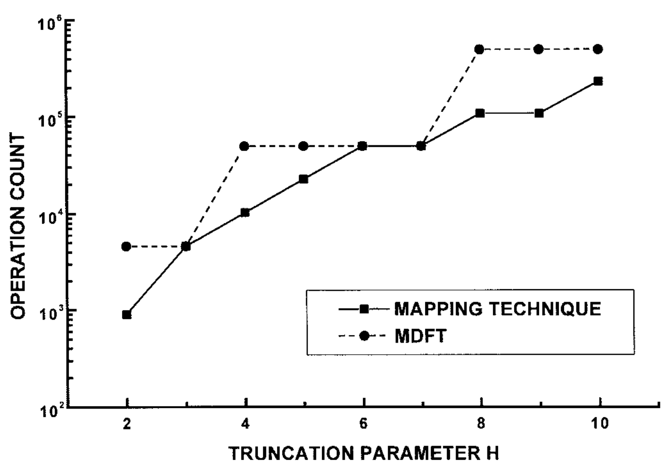

Fig. 2. Operation counts for the mapping technique and for the MDFT as functions of the truncation parameter H.

The operation count of the mapping technique is proportional to $S \log_2 S$, where S is the number of time-domain samples employed in the FFT. For the MDFT, the operation count is proportional to $S_1 S_2 S_3 \log_2(S_1 S_2 S_3)$, where S_1, S_2, and S_3 are the numbers of samples in each dimension. Recall that all these numbers must be powers of two. The operation counts of the two transforms as functions of H are in Fig. 2 and also on Table I. The mapping technique always has an operation count less than or equal to that of the MDFT.

V. CONCLUSION

A mapping technique suitable for the analysis of almost-periodic circuits with any number of fundamental frequencies has been presented. This technique is flexible and simple to implement since it employs a single one-dimensional FFT. The generality of the mapping technique associated with box truncation described in this letter makes this technique a good alternative for general-purpose HB simulators. Recall that the MDFT also requires box truncation, but mapping techniques are faster.

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